Optimal Execution under Adaptive Conjectures

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1 Introduction

In a strand of researches about the optimal execution problem, Ishii [17] introduced market participants other than the institutional investor and constructed a general equilibrium model. But the trade concentration on the final period takes place over a wide range of parameters in this model. In the case where the small investors are passive, the institutional investor will sell equally-divided amounts of securities at each period. Is it distant to assume the rational expectation about the others' orders placed off the equilibrium path ? One can think that it is natural to base the decision on thing of the past under similar circumstances.

2 Model

There are one institutional investor (that we call II in what follows) and a lot of small investors (SI). They can trade at times $t=1, 2, \cdots, T$. II has to sell W_1 units of security over this time period. Trades occur at $t=1, 2, \cdots, T$, and those who have one unit of securities obtain dividends $\tilde{F}$ at $t=T+1$.

$$\tilde{F}=F+\varepsilon, \tag{1}$$

where F is observed by all traders at $t=0$. ε follows a normal distribution that has a mean 0 and a variance σ_F^2 at $t=T+1$.

II places a market order S_t at every period $t=1, 2, \cdots, T$. II is risk neutral. That is, he maximizes

$$\sum_{t=1}^{T} P_t S_t \text{ s. t. } \sum_{t=1}^{T} S_t = W_1. \tag{2}$$

We define $W_{t+1}=W_t-S_t$ $(t=1, 2, \cdots, T)$.

There are infinitely many SIs. They are uniformly distributed whose population is 1. The measure of each SI is 0. SIs have no position at $t=0$. They can borrow some money or securities and place limit orders at $t=1, 2, \cdots, T$. They face no liquidity constraint. The interest rate is 0 for simplicity. We denote the quantity possessed by a representative SI[1] at the end of t as B_t. He places his orders "passively." That is, he maximizes

$$E[-\exp\{-\rho \sum_{u=1}^{t} (\tilde{F}-P_u)B_u\}|F], \tag{3}$$

at each period t.

The price P_t is determined to conform the amount of all buy order to all sell order at every $t=1, 2, \cdots, T$:

$$S_t=B_t. \tag{4}$$

3 Rational Equilibrium (Benchmark Model)

Prior to the consideration of II's evolutionary decision making, we will see a rational equilibrium, where II comprehend the inventory effect. Taking the result in advance, II even up W_1 and sell off equally:

$$S_t=\frac{W_1}{T} \text{ (for all } t). \tag{5}$$

3.1 Decision Making of SIs

The conditional expectation and the conditional variance that the representative SI has at $t\,(t=1, 2, \cdots, T)$ are

$$E[\sum_{u=1}^{t} (\tilde{F}-P_u)B_u|F]=\sum_{u=1}^{t} (F-P_u)B_u, \text{ and} \tag{6}$$

$$Var[\sum_{u=1}^{t} (\tilde{F}-P_u)B_u]=\sigma_F^2 (\sum_{u=1}^{t} B_u)^2. \tag{7}$$

The first order condition is

$$-\rho(F-P_t-\rho\sigma_F^2 \sum_{u=1}^{t} B_u)E[-\exp\{-\rho \sum_{u=1}^{t} (\tilde{F}-P_u)B_u\}|F]=0 \tag{8}$$

$$\Leftrightarrow B_t=\frac{F-P_t}{\rho\sigma_F^2}-\sum_{u=1}^{t-1} B_u. \tag{9}$$

Rearranging this term with the use of $B_t=S_t$, we obtain

$$P_t=F-\rho\sigma_F^2 \sum_{u=1}^{t} S_u. \tag{10}$$

4 Decision Making of II

4.1 Period T

By II's sellout constraint $S_T=W_T$

$$S_T = W_T. \tag{11}$$

By the market clearing condition,

$$P_T = F - \rho\sigma_F^2 \sum_{u=1}^{T-1} S_u - \rho\sigma_F^2 W_T. \tag{12}$$

Therefore the continuation selling amount is

$$V_T = (F - \rho\sigma_F^2 \sum_{u=1}^{T-1} S_u - \rho\sigma_F^2 W_T) W_T. \tag{13}$$

4.2 Induction Hypothesis of Period $t+1$ ($t=T-1, T-2, \cdots, 1$)

$$V_{t+1} = \left(F - \rho\sigma_F^2 \sum_{u=1}^{t} S_u - \frac{T-t+1}{2(T-t)} \rho\sigma_F^2 W_{t+1}\right) W_{t+1}. \tag{14}$$

4.3 Period t

The II's optimization problem at the period t is:

$$\max_{S_t} P_t S_t + V_{t+1}$$

$$= \max (F - \rho\sigma_F^2 \sum_{u=1}^{t} S_u) S_t + \left(F - \rho\sigma_F^2 \sum_{u=1}^{t} S_u - \frac{T-t+1}{2(T-t)} \rho\sigma_F^2 (W_t - S_t)\right)(W_t - S_t). \tag{15}$$

The first order condition is

$$\begin{aligned} & -\rho\sigma_F^2 S_t + F - \rho\sigma_F^2 \sum_{u=1}^{t} S_u \\ & + \left(-\rho\sigma_F^2 + \frac{T-t+1}{2(T-t)} \rho\sigma_F^2\right)(W_t - S_t) \\ & - \left(F - \rho\sigma_F^2 \sum_{u=1}^{t} S_u - \frac{T-t+1}{2(T-t)} \rho\sigma_F^2 (W_t - S_t)\right) \\ & = 0. \end{aligned} \tag{16}$$

Therefore

$$S_t = \frac{1}{T-t+1} W_t, \tag{17}$$

$$V_t = \left(F - \rho\sigma_F^2 \sum_{u=1}^{t-1} S_u - \frac{T-t+2}{2(T-t+1)} \rho\sigma_F^2 W_t\right) W_t. \tag{18}$$

Now we obtain the following results:

$$S_1 = S_2 = \cdots = S_T = \frac{W_1}{T}, \tag{19}$$

$$P_t = F - \frac{t}{T} \rho\sigma_F^2 W_1. \tag{20}$$

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5 Evolutionary Decision Making

We consider the evolutionary situation where II makes a decision on the execution problrem without any regard for the inventory effect.

Periodically II, SIs and the noise trader play the execution problem a number of times. The strategies of all traders are arbitrarily fixed. In each round, all traders change their strategies to the best responses under the condition in which every trader expects others' orders would be the same orders as the ones at the last round. For example, when II's market orders are $(S_t)_{t=1}^{T}$ at the round before the last round, then SIs' orders are

$$B_t=\frac{F-P_t}{\rho\sigma_F^2}-\sum_{u=1}^{t-1}S_u, \tag{21}$$

at the last round. Observing this, II's orders at the present round are

$$\arg\max_{(S'_t)_{t=1}^{T}}\sum_{t=1}^{T}\left(F-\rho\sigma_F^2\sum_{u=1}^{t-1}S_u-\rho\sigma_F^2S'_t\right)S'_t. \tag{22}$$

We are going to specify the absorbing state in this dynamic.

Theorem 1 *There is a unique absorbing state, where II executes more amount of securities in earlier period*:

$$S_t=\frac{2^{T-t}}{2^T-1}W_1,\ \text{and}$$

$$P_t=F-\frac{2^T-2^{T-t}}{2^T-1}\rho\sigma_F^2W_1, \tag{23}$$

$$\text{for all } t.$$

Proof. *Suppose $S_t>2S_{t+1}$ for some t ($t=1,2,\cdots,T-1$). Then II becomes better off than before by changing the orders at t and $t+1$ to $S'_t=S_t-\Delta$ and $S'_{t+1}=S_{t+1}+\Delta$ respectively for some small $\Delta>0$ and not changing the orders at other periods, which results in improvement in*

$$(S_t-2S_{t+1}-\Delta)\rho\sigma_F^2\Delta>0. \tag{24}$$

The same is equally true of the case $S_t<2S_{t+1}$. There exist selection pressures toward (23). *Meanwhile, suppose that* (23) *holds. If II changes his orders at the period t and u to $S'_t=S_t-\Delta$, $S'_u=S_u+\Delta$ and if he does not change the orders at other periods for any $t\neq u$, $\Delta\neq 0$, then he has to be worse off.*

$$-\rho\sigma_F^2\Delta^2<0. \tag{25}$$

6 Concluding Comments

In the case of evolutionary decision making, the faster trading takes place. The prices bocome lower.

Note

1 When all SIs place the identical orders, the aggregate SIs' order is the same as in the case where one trader with the risk aversion ϱ places his order. So we can call this virtual tarder a "representative SI."

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